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Unique Local Solutions for a 3D Keller – Segel – Navier – Stokes System with Nonlinear Diffusion and Logistic Source

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Abstract

In this paper, we are concerned with the Cauchy problem of the chemotaxis Keller-Segel-Navier-Stokes system with a logistic source in three-dimensions. We explore mathematical representations that capture the behavior of oxygen, swimming bacteria, and viscous, incompressible fluids. The local existence and uniqueness of solution for this system is established by the classical energy method combined with compactness criteria and the damping effect of the logistic source.

Keywords: Chemotaxis, Attraction–repulsion, Logistic source, Energy method.

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حلول محلية فريدة لنظام كيلر – سيجل– نافييه – ستوكس ثلاثي الأبعاد مع الانتشار غير الخطي والمصدر اللوجستي

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الملخص

في هذه الورقة البحثية، نتناول مسألة كوشي لنظام التاكسي الكيميائي – كيلر– سيجل –نافييه – ستوكس ذي المصدر اللوجيستي ثلاثي الأبعاد. نستكشف تمثيلات رياضية ترصد سلوك الأكسجين، والبكتيريا السابحة، والسوائل اللزجة غير القابلة للانضغاط. يتم إثبات الوجود المحلي وتفرد الحل لهذا النظام من خلال طريقة الطاقة الكلاسيكية جنبًا إلى جنب مع معايير الاكتناز وتأثير التخميد للمصدر اللوجستي.

الكلمات المفتاحية: التاكسي الكيميائي، الجذب والتنافر، المصدر اللوجستي، طريقة الطاقة.

Introduction

Chemotaxis refers to the directed movement of cells or organisms in response to chemical gradients. It describes many biological phenomena, such as the interaction between cells and the environment and various biological processes, as well as bacteria's sensing of chemicals in the environment through receptor proteins and the regulation of flagella to achieve Chemotaxis. The chemotaxis model was first introduced by (Keller, Segel, 26(1970)), (Keller, Segel, 30(1971)), as follows:

$$\begin{aligned} n_t &= \Delta n - \nabla.(n\nabla c), \quad x\in\Omega, t>0,\\ c_t &= \Delta c - c + n, \qquad x\in\Omega, t>0. \end{aligned} \tag{1.1}$$

Here $n(x,t) and\ c(x,t)$ represent the density of the bacteria and oxygen concentration at position x and $t>0$, respectively. Chemotactic flux, Δn denotes migration of the bacteria, Δc is the diffusion of chemoattractant. The chemotaxis model (1.1) and its variants have been widely studied by many authors. Tao [3] studied the condition for the global existence of a classical solution

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to the system (1.1) for $n \geq 2$. Zhang and Li [4] proved the global existence of a classical solution in two spatial dimensions. Nie and Zheng [5] studied the global well-posedness in the two-dimensional case where the damping effect of the logistic source was used. The mathematical analysis of (1.1) and its variant forms mainly focuses on the boundedness and blow-up of the solutions. For related work, we can refer to ([6], [7], [8], [9], [10]), and fruitful results have been achieved in the global existence and uniform boundedness of solutions. When the space dimension, all solutions are global and bounded. It is well-known that for large classes of initial data, solutions of system (1.1) blow up when the space dimension and the total mass of cells are large, while global bounded solutions can be generated under small conditions on the initial data ([11], [12], [13], [14], [15], [16], [17], [18], [19]). For the more related works in this direction, we mention that a corresponding quasilinear version has been deeply investigated by ([20], [21], [22], [23], [24]).
In recent years, the chemotaxis model that has received the most attention in study has been the classic Keller-Segel system with indirect signal production:

$$\begin{aligned} &u_t = \Delta u - \nabla.(u\nabla v) + \kappa u - \mu u^{\gamma}, && x \in \Omega, t > 0, \\ &v_t = \Delta v - v + u, && x \in \Omega, t > 0. \qquad (1.2) \end{aligned}$$

The model (1.2) describes a part of cellular slime molds with the chemotaxis at the life cycle. Where u and v respectively denote the population density of cells and the signal concentration of chemical, with a smooth boundary $\partial\Omega$. Here, $\kappa \in \mathbb{R}$, $\mu > 0$, $\gamma > 0$ are parameters reflecting proliferation and death of cells in a logistic law, $+u$ represents the spontaneous production of the attractant and is proportional to the number of amoebae u, while $-v$ represents decay of attractant activity. More precisely, Winkler [25] derived that system (1.2) admits a unique global-in-time bounded classical solution if μ is large enough and initial data is sufficiently smooth. Subsequently, a large body of works is devoted to studying qualitatively and quantitatively on μ when logistic damping suppresses bacteria aggregation. The model (1.2) admits a global bounded classical solution in $\mathbb{R}^1$, as demonstrated

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by Osaki and Yagi [26]. They also derived similar results in $\mathbb{R}^2$ [27, 28] in 2002. Furthermore, the chemotaxis is described by related mathematical models. Hattori and Lagha [29, 30] showed the global existence and asymptotic behavior of the solutions for a chemotaxis system with chemoattractant and repellent in three dimensions. Duan, Liu, and Zhu [31] proved the time-decay rate by the combination of energy estimates and spectral analysis. In this paper, we are concerned with the following initial value problem of the Keller-Segel-Navier-Stokes system with nonlinear diffusion:

$$\partial_t n + u.\nabla n = \Delta n - \chi\nabla.(\mathrm{n}\nabla c) - \rho n + \tau n^2$$

$$\partial_t u + u.\nabla u + \nabla \pi = \Delta u - n\nabla\varphi$$

$$\partial_t c + u.\nabla c = \Delta c - nc + \alpha c$$

$$\partial_t v + u.\nabla v = \Delta v + nv - \gamma v$$

$$\nabla.u = 0 \quad t > 0, \qquad x \in \mathbb{R}^3. \qquad (1.3)$$

With initial data

$$(n,u,c,v)|_{t=0} = \big(n_0(x), u_0(x), c_0(x), v_0(x)\big),\ x \in \mathbb{R}^3, \quad (1.4)$$

where $\big(n_0(x), u_0(x), c_0(x), v_0(x)\big) \to (0{,}0{,}0{,}0)$ as $|x| \to \infty$. We begin with a brief description of the system (1.3), which characterizes the interaction between cells, their fluid environment, and chemicals contained therein. Here, $n = n(t,x)$, $c = c(t,x), and\ u = u(t,x)$ denote the cell concentration, the oxygen concentration, and the fluid velocity field of the surrounding environment respectively. In addition, $\pi = \pi(t,x)$ is unknown pressure, and φ is the gravitational potential function. The chemotaxis term $-\chi\nabla.(\mathrm{n}\nabla c)$ shows that cells are moving towards a high concentration of oxygen. While $n_0 = n_0(x)$, $u_0 = u_0(x)$, $c_0 = c_0(x)$, and $v_0 = v_0(x)$ are the given functions, the constants $\rho > 0$, $\tau > 0, \gamma > 0$, $\chi > 0$. A simple model case can be obtained upon the choices $\nabla\varphi = const.$, $\chi = 1$.

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This paper is organized as follows: In section 2, we present notations and some assumptions that will be used throughout the whole paper and state our main result. In Section 3, we prove the local-in-time existence of solution for a three-dimensional chemotaxis system with incompressible Keller-Segel-Navier-Stokes equations.

2. Main result

We first introduce some notations that we will use later in this paper. Let $1 \leq P \leq \infty$, we denote L^p for the Lebesgue space on Ω, and the norms in the space $L^p(\mathbb{R}^3)$ are denoted by $\|.\|_{\mathrm{p}}$. For nonnegative integer K, we denote by H^{k} the Sobolev space of order K. Set $\mathrm{L}^2 = \mathrm{H}^0$, the norm of H^{k} is denoted by $\|.\|_{\mathrm{H}^{\mathrm{k}}}$. We set $\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ for a multi-index $\alpha = [\alpha_1, \alpha_2, \alpha_3]$ and length of α is $|.| = \alpha_3 + \alpha_2 + \alpha_3$. C and $\mathrm{C_i}$, where $i = 1,2$, denote some positive (generally small) constant, where both C and $\mathrm{C_i}$ may take different values in different places. Let us denote the space

$$X(0,T) = \{(n,u,c,v) \in \mathrm{C}\big([0,\mathrm{T}];\mathrm{H}^3(\mathbb{R}^3)\big) \cap \mathrm{C}^1\big([0,\mathrm{T}];\mathrm{H}^1(\mathbb{R}^3)\big),$$
$$(\nabla n, \nabla u, \nabla \mathrm{c}, \nabla v) \in \mathrm{L}^2\big([0,\mathrm{T}];\mathrm{H}^3(\mathbb{R}^3)\big)\}.$$

The main goal of this paper is to establish the existence of unique local solutions in three dimensions for the above system (1.3). The main result of this paper is stated as follows.

Theorem 2.1. There exists a positive number ε_0 such that if

$$\|n_0, u_0, c_0, v_0\|_{\mathrm{H}^3} \leq \varepsilon_0,$$

then the Cauchy problems (1.3)-(1.4) of the Keller-Segel-Navier-Stokes system admits a unique local solution (n,u,c,v) with

$$(n,u,c,v) \in X(0,T).$$

The proof of the existence of local solutions in Theorem 2.1 by constructing a sequence of approximation functions based on iteration and some basic energy estimates. Let $\mathrm{W}(t) = (n,u,c,v)$

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be a smooth solution to the Cauchy problem of the chemotaxis system (1.4) with initial data $W_0 = (n_0, u_0, c_0, v_0)$.

3. Existence of local solutions

In this section, we show the proof of Theorem 2.1. We construct the sequence $(n^j, u^j, c^j, v^j)_{j\geq 0}$ by iteratively solving the Cauchy problem on the following:

$$\begin{cases} \partial_t n^{j+1} - \Delta n^{j+1} + \rho n^{j+1} = -u^j.\nabla n^{j+1} - \nabla.\left(n^j \nabla c^{j+1}\right) + \tau n^{j2} \\ \partial_t u^{j+1} - \Delta u^{j+1} + \nabla \pi^{j+1} = -u^j.\nabla u^{j+1} - n^j \nabla \varphi \\ \partial_t c^{j+1} - \Delta c^{j+1} + \alpha c^{j+1} = -u^j.\nabla c^{j+1} - n^j c^{j+1} \\ \partial_t v^{j+1} - \Delta v^{j+1} + \gamma v^{j+1} = -u^j.\nabla v^{j+1} + n^j v^{j+1} \\ \nabla.u^{j+1} = 0 \quad t > 0, \quad x \in \mathbb{R}^3, \end{cases} \tag{3.1}$$

With initial data

$$\left(n^{j+1}, u^{j+1}, c^{j+1}, v^{j+1}\right)\big|_{t=0} = (n_0, u_0, c_0, v_0), \qquad x \in \mathbb{R}^3, \tag{3.2}$$

for $j \geq 0$, where $(n^0, u^0, c^0, v^0)|_{t=0} \to (0,0,0,0)$ as $|x| \to \infty$, is set at the initial step. For simplicity, In what follows, we write $W^j = \left(n^j, u^j, c^j, v^j\right)_{j\geq 0}$ and $W_0 = (n_0, u_0, c_0, v_0)$, where $W^0 \equiv (0,0,0,0)$. Next, we prove that $(W^j)_{j\geq 0}$ is a Cauchy sequence in the Banach space $C([0,T_1];H^3)$ for $T_1 > 0$ suitable small. At last, by taking the limit and continuous argument, we prove that (n, u, c, v) is a local solution to (1.3)-(1.4). Now, we can state the following result:

Theorem 3.1.

There are constants $\varepsilon_1 > 0$, $T_1 > 0$, $M > 0$ such that if the initial data $W_0 \in H^3(\mathbb{R}^3)$ and $\|W_0\|_{H^3} \leq \varepsilon_1$, then for each $j \geq 0$, $W^j \in C([0, T_1)]; H^3)$ is well defined and

$$\sup_{0\leq t\leq T_1} \left\|W^j(t)\right\|_{H^3} \leq M. \tag{3.3}$$

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Moreover, $(W^j)_{j\geq 0}$ is a Cauchy sequence in Banach space $C([0, T_1];\ H^3)$, the corresponding limit function denoted by W belongs to $C([0, T_1]; H^3)$ with

$$\sup_{0\leq t\leq T_1} \|W(t)\|_{H^3} \leq M\,, \qquad (3.4)$$

and $W = (n, u, c, v)$ is a solution to the Cauchy problem (1.3)-(1.4) over $[0, T_1]$. The Cauchy problem (1.3)-(1.4) admits at most one solution $W\in C([0, T_1]; H^3)$, which satisfies (3.4).

proof

We use the induction to prove (3.3). The trivial case is $j = 0$ since by the assumption at the initial step. Suppose that (3.3) holds for some $j \geq 0$ where M is small enough. To prove (3.3) holds for $j + 1$, we need some energy estimates on $(n^{j+1}, u^{j+1}, c^{j+1}, v^{j+1})$.

Applying ∂^α to both sides of the first equation of (3.1), multiplying by $\partial^\alpha n^{j+1}$, and integrating over $\mathbb{R}^3$ with $|\alpha| \leq 3$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left|\partial^\alpha n^{j+1}\right|^2 dx - \int_{\mathbb{R}^3}\partial^\alpha \Delta n^{j+1}\,\partial^\alpha n^{j+1} dx$$
$$+\rho\int_{\mathbb{R}^3}\left|\partial^\alpha n^{j+1}\right|^2 dx = -\int_{\mathbb{R}^3}\partial^\alpha\left(u^j.\nabla n^{j+1}\right)\partial^\alpha n^{j+1}\,dx$$
$$-\int_{\mathbb{R}^3}\partial^\alpha\nabla.\left(n^j\nabla c^{j+1}\right)\partial^\alpha n^{j+1}\,dx\ +\tau\int_{\mathbb{R}^3}\partial^\alpha n^{j+1}\,\partial^\alpha n^{j2}\ dx. \quad (3.5)$$

By using integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left|\partial^\alpha n^{j+1}\right|^2 dx + \int_{\mathbb{R}^3}\left|\partial^\alpha \nabla n^{j+1}\right|^2 dx + \rho\int_{\mathbb{R}^3}\left|\partial^\alpha n^{j+1}\right|^2 dx,$$
$$= -\int_{\mathbb{R}^3}\partial^\alpha\left(u^j.\nabla n^{j+1}\right)\partial^\alpha n^{j+1}\,dx - \int_{\mathbb{R}^3}\partial^\alpha\left(n^j\nabla c^{j+1}\right)\partial^\alpha\nabla n^{j+1}\,dx$$
$$+\tau\int_{\mathbb{R}^3}\partial^\alpha n^{j+1}\,\partial^\alpha n^{j2}\ dx. \qquad (3.6)$$

Where the terms on the right-hand side are bounded by

$$C\left\|u^j\right\|_{H^3}\left\|\nabla n^{j+1}\right\|_{H^3}^2 + C\left\|n^j\right\|_{H^3}\left\|\nabla c^{j+1}\right\|_{H^3}\ \left\|\nabla n^{j+1}\right\|_{H^3}$$
$$+\ C\left\|n^j\right\|_{H^2}\ \left\|\nabla n^{j+1}\right\|_{H^3}\left\|n^j\right\|_{H^3}.$$

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Then, after taking the summation over $|\alpha| \leq 3$ and using the Cauchy inequality, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\|n^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2+\frac{1}{2}\left\|\nabla n^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2+\rho\left\|n^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2$$
$$\leq \mathrm{C}\left\|(\mathrm{u}^{\mathrm{j}}, n^{\mathrm{j}})\right\|_{\mathrm{H}^3}^2\left\|\nabla\left(n^{\mathrm{j}+1}, \mathrm{c}^{\mathrm{j}+1}\right)\right\|_{\mathrm{H}^3}^2+C\left\|n^{\mathrm{j}}\right\|_{\mathrm{H}^3}^2. \qquad (3.7)$$

By the same way, for $(3.1)_2$ on $\mathrm{u}^{\mathrm{j}+1}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^3}\left|\partial^\alpha \mathrm{u}^{\mathrm{j}+1}\right|^2 \mathrm{dx}+\int_{\mathbb{R}^3}\left|\nabla\,\partial^\alpha \mathrm{u}^{\mathrm{j}+1}\right|^2 \mathrm{dx}=$$
$$-\int_{\mathbb{R}^3}\partial^\alpha\nabla\pi^{j+1}.\,\partial^\alpha \mathrm{u}^{\mathrm{j}+1}\mathrm{dx}-\int_{\mathbb{R}^3}\partial^\alpha\left(\mathrm{u}^{\mathrm{j}}.\,\nabla \mathrm{u}^{\mathrm{j}+1}\right).\,\partial^\alpha \mathrm{u}^{\mathrm{j}+1}\mathrm{dx}$$
$$+\int_{\mathbb{R}^3}\partial^\alpha\left(\nabla n^{\mathrm{j}}\varphi\right).\,\partial^\alpha \mathrm{u}^{\mathrm{j}+1}\mathrm{dx}\,. \qquad (3.8)$$

By using the Cauchy inequality the right-hand side of the previous equation is bounded by

$$\mathrm{C}\left\|u^{\mathrm{j}}\right\|_{\mathrm{H}^3}^2\left\|\nabla \mathrm{u}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2+\mathrm{C}\left\|n^{\mathrm{j}}\right\|_{\mathrm{H}^3}^2+\frac{1}{2}\left\|\nabla \mathrm{u}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2.$$

Then, after taking summation over $|\alpha| \leq 3$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\|\mathrm{u}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2+\frac{1}{2}\left\|\nabla \mathrm{u}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2$$
$$\leq \mathrm{C}\left\|n^{\mathrm{j}}\right\|_{\mathrm{H}^3}^2+\mathrm{C}\left\|u^{\mathrm{j}}\right\|_{\mathrm{H}^3}^2\left\|\nabla \mathrm{u}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2. \qquad (3.9)$$

By the same way, for $(3.1)_3$ on $\mathrm{c}^{\mathrm{j}+1}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^3}\left|\partial^\alpha \mathrm{c}^{\mathrm{j}+1}\right|^2\mathrm{dx}+\int_{\mathbb{R}^3}\left|\nabla\,\partial^\alpha \mathrm{c}^{\mathrm{j}+1}\right|^2\mathrm{dx}+\alpha\int_{\mathbb{R}^3}\left|\partial^\alpha \mathrm{c}^{\mathrm{j}+1}\right|^2\mathrm{dx}$$
$$=-\int_{\mathbb{R}^3}\partial^\alpha(\mathrm{u}^{\mathrm{j}}.\,\nabla \mathrm{c}^{\mathrm{j}+1})\,\partial^\alpha \mathrm{c}^{\mathrm{j}+1}\mathrm{dx}-\int_{\mathbb{R}^3}\partial^\alpha\left(n^{\mathrm{j}}\mathrm{c}^{\mathrm{j}+1}\right)\partial^\alpha \mathrm{c}^{\mathrm{j}+1}\mathrm{dx}\,. \qquad (3.10)$$

The terms on the right-hand side of the previous inequality are bounded by

$$\mathrm{C}\left\|\mathrm{u}^{\mathrm{j}}\right\|_{\mathrm{H}^3}\left\|\nabla \mathrm{c}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2+\mathrm{C}\left\|n^{\mathrm{j}}\right\|_{\mathrm{H}^3}\left\|\nabla \mathrm{c}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}\left\|\mathrm{c}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}.$$

By using the Cauchy inequality, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\|\mathrm{c}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2+\frac{1}{2}\left\|\nabla \mathrm{c}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2+\frac{\alpha}{2}\left\|\mathrm{c}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2$$
$$\leq \mathrm{C}\left\|(\mathrm{u}^{\mathrm{j}}, n^{\mathrm{j}})\right\|_{\mathrm{H}^3}^2\left\|\nabla \mathrm{c}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2. \qquad (3.11)$$

By the same way, for $(3.1)_4$ on $v^{\mathrm{j}+1}$, we obtain

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$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\left|\partial^\alpha v^{j+1}\right|^2 dx + \int_{\mathbb{R}^3}\left|\nabla\,\partial^\alpha v^{j+1}\right|^2 dx \quad +\gamma\int_{\mathbb{R}^3}\left|\partial^\alpha v^{j+1}\right|^2 dx$$
$$= -\int_{\mathbb{R}^3}\partial^\alpha(u^j.\nabla v^{j+1})\,\partial^\alpha v^{j+1}dx$$
$$-\int_{\mathbb{R}^3}\partial^\alpha\left(n^j v^{j+1}\right)\partial^\alpha v^{j+1}dx. \qquad (3.12)$$

The terms on the right-hand side of the previous inequality are bounded by

$$C\left\|u^j\right\|_{H^3}\left\|\nabla v^{j+1}\right\|_{H^3}^2 + C\left\|n^j\right\|_{H^3}\left\|\nabla v^{j+1}\right\|_{H^3}\left\|v^{j+1}\right\|_{H^3}.$$

Thus, we get

$$\frac{1}{2}\frac{d}{dt}\left\|v^{j+1}\right\|_{H^3}^2 + \frac{1}{2}\left\|\nabla v^{j+1}\right\|_{H^3}^2 + \frac{\gamma}{2}\left\|v^{j+1}\right\|_{H^3}^2$$
$$\le C\left\|(u^j, n^j)\right\|_{H^3}^2\left\|\nabla v^{j+1}\right\|_{H^3}^2. \qquad (3.13)$$

By taking the linear combination of inequalities (3.7),(3.9),(3.11),(3.13) leads to

$$\frac{1}{2}\frac{d}{dt}\left(\left\|n^{j+1}\right\|_{H^3}^2 + \left\|u^{j+1}\right\|_{H^3}^2 + \left\|c^{j+1}\right\|_{H^3}^2 + \left\|v^{j+1}\right\|_{H^3}^2\right)$$
$$+C_1\left\|\nabla(n^{j+1}, u^{j+1}, c^{j+1}, v^{j+1})\right\|_{H^3}^2 + C_2\left\|n^{j+1}, c^{j+1}, v^{j+1}\right\|_{H^3}^2$$
$$\le C\left\|(n^j, u^j)\right\|_{H^3}^2$$
$$+ C\left\|\left(u^j, n^j\right)\right\|_{H^3}^2\left\|\nabla(n^{j+1}, u^{j+1}, c^{j+1}, v^{j+1})\right\|_{H^3}^2. \qquad (3.14)$$

Thus, after integrating (3.14) over $[0, t]$ for all t $\in[0, T_1]$, we have

$$\left\|W^{j+1}(t)\right\|_{H^3}^2 + C_1\int_0^t\left\|\nabla W^{j+1}(s)\right\|_{H^3}^2 ds + C_2\int_0^t\left\|n^{j+1}, c^{j+1}, v^{j+1}\right\|_{H^3}^2$$
$$\le C\left\|W_0\right\|_{H^3}^2 + C\int_0^t\left\|W^j(s)\right\|_{H^3}^2 ds$$
$$+C\int_0^t\left\|W^j(s)\right\|_{H^3}^2\left\|\nabla W^{j+1}(s)\right\|_{H^3}^2 ds, \qquad (3.15)$$

for some positive constants C_1 *and* C_2. From the inductive assumption, the previous inequality can be re-estimated as

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$$\left\|W^{j+1}(t)\right\|_{H^3}^2 + C_1\int_0^t \left\|\nabla W^{j+1}(s)\right\|_{H^3}^2 ds$$
$$+ C_2\int_0^t \left\|n^{j+1}, c^{j+1}, v^{j+1}\right\|_{H^3}^2 ds$$
$$\leq C\varepsilon_1^2 + CM^2T_1 + CM^2\int_0^t \left\|\nabla W^{j+1}(s)\right\|_{H^3}^2 ds, \qquad (3.16)$$

for any $0 \leq t \leq T_1$. Now, we take the small constants $\varepsilon_1 > 0$, $M > 0$ and $T_1 > 0$. Then, we obtain

$$\left\|W^{j+1}(t)\right\|_{H^3}^2 + C_1\int_0^t \left\|\nabla W^{j+1}(s)\right\|_{H^3}^2 ds + C_2\int_0^t \left\|n^{j+1}, c^{j+1}, v^{j+1}\right\|_{H^3}^2 ds \leq M^2, \qquad (3.17)$$

for any $0 \leq t \leq T_1$. This implies that (3.3) holds for $j+1$, Hence (3.3) is proved for all $j \geq 0$.

Next, we define the equivalent energy functional

$$E(W^{j+1}(t)) := \left\|n^{j+1}\right\|_{H^3}^2 + \left\|u^{j+1}\right\|_{H^3}^2 + \left\|c^{j+1}\right\|_{H^3}^2 + \left\|v^{j+1}\right\|_{H^3}^2.$$

Similar to prove (3.17), we have

$$\left|E(W^{j+1}(t)) - E(W^{j+1}(s))\right| = \left|\int_s^t \frac{d}{d\tau} E(W^{j+1}(\tau))\, d\tau\right|$$
$$\leq C\int_s^t \left\|W^j(\tau)\right\|_{H^3}^2 d\tau + + C\int_0^t \left\|n^{j+1}, c^{j+1}, v^{j+1}\right\|_{H^3}^2 d\tau + C\int_s^t \left(1 + \left\|W^j(\tau)\right\|_{H^3}^2\right)\left\|\nabla W^{j+1}(\tau)\right\|_{H^3}^2 d\tau$$
$$\leq CM^2(t-s) + C(1+M^2)\int_s^t \left\|\nabla W^{j+1}(\tau)\right\|_{H^3}^2 d\tau \quad (3.18)$$

for any $0 \leq s \leq t \leq T_1$. Here, The time integral on the right-hand side from the above inequality is bounded by (3.17), and hence $E(W^{j+1}(t))$ is Continuous in t for each $j \geq 0$. By the same argument, we can infer that $\left\|u^{j+1}\right\|_{H^3}^2$, $\left\|c^{j+1}\right\|_{H^3}^2$ and $\left\|v^{j+1}\right\|_{H^3}^2$ are continuous in t. From the continuity of $E(W^{j+1}(t))$, we can also infer the continuity of $\left\|n^{j+1}\right\|_{H^3}^2$. Therefore, $\left\|W^{j+1}(t)\right\|_{H^3}^2$ is continuous in time for any $j \geq 1$.

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Now, we prove that the sequence $(W^j)_{j\geq 0}$ is a Cauchy sequence in the Banach space $C([0,T_1];H^3)$, which converges to the solution $W=(n,u,c,v)$ of the Cauchy problem (3.1)-(3.2), and satisfies (3.4). For simplicity, we denote $\delta f^{j+1}=f^{j+1}-f^j$. Taking the difference of (3.1) for $j+1$ and j so that it gives:

$$\begin{aligned}
&\partial_t\delta n^{j+1}-\Delta\delta n^{j+1}+\rho\delta n^{j+1}=-u^j.\nabla\delta n^{j+1}-\delta u^j.\nabla n^j\\
&\qquad\qquad -\nabla.\left(n^j\nabla\delta c^{j+1}\right)-\nabla.\left(\delta n^j\nabla c^j\right)\ -\tau\delta n^{j2}\\
&\partial_t\delta u^{j+1}-\Delta\delta u^{j+1}\\
&\qquad\qquad =\ -\nabla\delta\pi^{j+1}-u^j.\nabla\delta u^{j+1}-\delta u^j.\nabla u^j-\delta n^j\nabla\varphi\\
&\partial_t\delta c^{j+1}-\Delta\delta c^{j+1}+\alpha\delta c^{j+1}\\
&\qquad\qquad =-u^j.\nabla\delta c^{j+1}-\delta u^j.\nabla c^j-n^j\delta c^{j+1}-\delta n^j c^j,\\
&\partial_t\delta v^{j+1}-\Delta\delta v^{j+1}+\gamma\delta v^{j+1}\ =\\
&-u^j.\nabla\delta v^{j+1}-\delta u^j.\nabla v^j+\ -n^j\delta v^{j+1}-\delta n^j v^j,\\
&\nabla.\delta u^{j+1}=0,\qquad t>0,x\in\mathbb{R}^3. \qquad (3.19)
\end{aligned}$$

By using the same energy estimates as before, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\delta n^{j+1}\right\|_{H^3}^2+\left\|\nabla\delta n^{j+1}\right\|_{H^3}^2+\frac{\rho}{2}\left\|\delta n^{j+1}\right\|_{H^3}^2\\
&\leq C\left\|\left(\delta u^j,\delta n^j\right)\right\|_{H^3}^2\left\|\nabla\left(n^j,c^j\right)\right\|_{H^3}^2\\
&+C\left\|\left(n^j,u^j\right)\right\|_{H^3}^2\left\|\left(\delta n^{j+1},\delta c^{j+1}\right)\right\|_{H^3}^2\\
&+C\left\|\nabla\delta c^{j+1}\right\|_{H^3}^2 \qquad (3.20)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\delta u^{j+1}\right\|_{H^3}^2+\frac{1}{2}\left\|\nabla\delta u^{j+1}\right\|_{H^3}^2\\
&\leq C\left\|u^j\right\|_{H^3}^2\left\|\nabla\delta u^{j+1}\right\|_{H^3}^2\\
&+C\left\|\delta u^j\right\|_{H^3}^2\left\|\nabla u^j\right\|_{H^3}^2+C\left\|\delta u^j\right\|_{H^3}^2 \qquad (3.21)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\delta c^{j+1}\right\|_{H^3}^2+C\left\|\delta c^{j+1}\right\|_{H^3}^2+C\left\|\nabla\delta c^{j+1}\right\|_{H^3}^2\leq\\
&C\left\|\left(n^j,u^j\right)\right\|_{H^3}^2\left\|\nabla\delta c^{j+1}\right\|_{H^3}^2+C\left\|\left(\delta n^j,\delta u^j\right)\right\|_{H^3}^2\left\|\nabla c^j\right\|_{H^3}^2 \qquad (3.22)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\delta v^{j+1}\right\|_{H^3}^2+\frac{\gamma}{2}\left\|\delta v^{j+1}\right\|_{H^3}^2+\frac{1}{2}\left\|\nabla\delta v^{j+1}\right\|_{H^3}^2\leq\\
&C\left\|\left(n^j,u^j\right)\right\|_{H^3}^2\left\|\nabla\delta v^{j+1}\right\|_{H^3}^2+C\left\|\left(\delta n^j,\delta u^j\right)\right\|_{H^3}^2\left\|\nabla v^j\right\|_{H^3}^2. \qquad (3.23)
\end{aligned}$$

We combine the equations (3.20)-(3.23) to get

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$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{E}\left\|\delta \mathrm{W}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2+\mathrm{C}\left\|\nabla\delta \mathrm{W}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2+\mathrm{C}\left\|\delta n^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2$$

$$+\mathrm{C}\ \left\|\delta \mathrm{c}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2+\mathrm{C}\left\|\delta v^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2$$

$$\leq \mathrm{C}\left\|\delta \mathrm{W}^{\mathrm{j}}\right\|_{\mathrm{H}^3}^2+\mathrm{C}\left\|\delta \mathrm{W}^{\mathrm{j}}\right\|_{\mathrm{H}^3}^2\ \left\|\nabla \mathrm{W}^{\mathrm{j}}\right\|_{\mathrm{H}^3}^2$$

$$+\mathrm{C}\left\|\mathrm{W}^{\mathrm{j}}\right\|_{\mathrm{H}^3}^2\left\|\nabla\delta \mathrm{W}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2. \qquad (3.24)$$

By integrating over $[0,t]$, we obtain

$$\left\|\delta \mathrm{W}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2+C_1\int_0^t\left\|\nabla\delta \mathrm{W}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2 ds$$

$$+C_2\int_0^t\left\|\delta n^{\mathrm{j}+1},\delta \mathrm{c}^{\mathrm{j}+1}\delta v^{\mathrm{j}+1})\right\|_{\mathrm{H}^3}^2 ds$$

$$\leq\ C(1+\mathrm{M}^2)T_1\sup_{0\leq \mathrm{t}\leq \mathrm{T}_1}\left\|\delta \mathrm{W}^{\mathrm{j}}\right\|_{\mathrm{H}^3}^2+C\mathrm{M}^2\int_0^t\left\|\nabla\delta \mathrm{W}^{\mathrm{j}+1}\right\|_{\mathrm{H}^3}^2 ds,$$

for any $0\leq \mathrm{t}\leq \mathrm{T}_1$, which by smallness of M and T_1 implies that there is a constant $\theta\in(0,1)$ such that

$$\sup_{0\leq \mathrm{t}\leq \mathrm{T}_1}\left\|\delta \mathrm{W}^{\mathrm{j}+1}\right\|_{H^3}\leq\theta\sup_{0\leq \mathrm{t}\leq \mathrm{T}_1}\left\|\delta \mathrm{W}^{\mathrm{j}}\right\|_{H^3},$$

for any $\mathrm{j}\geq 1$. It can be seen from the above inequality that $\left(\mathrm{W}^{\mathrm{j}}\right)_{\mathrm{j}\geq 0}$ is a Cauchy sequence in the Banach space $\mathrm{C}([0,\mathrm{T}_1];\mathrm{H}^3)$. Thus the limit function

$$\mathrm{W}=\mathrm{W}^0+\lim_{\mathrm{i}\to\infty}\sum_{j=0}^{i}\left(\mathrm{W}^{\mathrm{j}+1}-\mathrm{W}^{\mathrm{j}}\right)$$

exists in $\mathrm{C}\big([0,\mathrm{T}_1];\mathrm{H}^3(\mathbb{R}^3)\big)$, and satisfies

$$\sup_{0\leq \mathrm{t}\leq \mathrm{T}_1}\|W(t)\|_{H^3}\leq\sup_{0\leq \mathrm{t}\leq \mathrm{T}_1}\lim_{j\to\infty} inf\left\|W^j(t)\right\|_{H^3}\leq \mathrm{M}. \qquad (3.25)$$

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Finally, we show that the Cauchy problem (1.3)-(1.4) admits at most one solution in $C([0,T_1];H^3(\mathbb{R}^3))$. Suppose that there exist two solutions $W,\widetilde{W}$ in $C([0,T_1];H^3(\mathbb{R}^3))$ which satisfy (1.4). Let $\tilde{n}=n_1(x,t)-n_2(x,t)$, $\tilde{u}=u_1(x,t)-u_2(x,t)$, $\tilde{c}=c_1(x,t)-c_2(x,t)$, and $\tilde{v}=v_1(x,t)-v_2(x,t)$
solves

$$\begin{aligned}
&\partial_t\,\tilde{n}-\Delta\tilde{n}+\rho\tilde{n}=-u_1.\nabla\tilde{n}-\tilde{u}.\nabla n_2+\nabla.(n_2\,\nabla\tilde{c})\\
&\qquad -\nabla.(\tilde{n}\nabla c_1)-\tau(n_1+n_2)\tilde{n}\\
&\partial_t\,\tilde{u}-\Delta\tilde{u}+\nabla\tilde{\pi}=-\,u_2.\nabla\tilde{u}-\tilde{u}\nabla\cdot u_1-\tilde{n}\nabla\varphi\\
&\partial_t\,\tilde{c}-\Delta\tilde{c}+\alpha\tilde{c}=-u_2.\nabla\tilde{c}-\tilde{u}.\nabla c_1+\tilde{n}c_2+n_1\tilde{c}\\
&\partial_t\,\tilde{v}-\Delta\tilde{v}+\gamma\tilde{v}=-u_2.\nabla\tilde{v}\\
&\qquad\qquad -\tilde{u}.\nabla v_1+\tilde{n}v_2+n_1\tilde{v}.
\end{aligned}\tag{3.26}$$

Multiplying $\tilde{n}$ to both sides of the first equation of (3.26) and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}\tilde{n}\,\partial_t\tilde{n}dx-\int_{\mathbb{R}^3}\tilde{n}\,\Delta\tilde{n}dx\\
&+\rho\int_{\mathbb{R}^3}\tilde{n}\,\tilde{n}dx\\
&=-\int_{\mathbb{R}^3}\tilde{n}\;u_1.\nabla\tilde{n}dx-\int_{\mathbb{R}^3}\tilde{n}\;\tilde{u}.\nabla n_2dx\\
&-\int_{\mathbb{R}^3}\tilde{n}\,\nabla.(n_2\nabla\tilde{c})dx\;-\int_{\mathbb{R}^3}\tilde{n}\,\nabla.(\tilde{n}\nabla c_1)dx\\
&-\tau\int_{\mathbb{R}^3}\tilde{n}\,(n_1+n_2)\tilde{n}dx.
\end{aligned}$$

Then, after using integration by parts and the Cauchy inequality, we obtain

$$\begin{aligned}
&\frac{d}{2\,dt}\|\tilde{n}\|_{L^2}^2+\rho\|\tilde{n}\|_{L^2}^2+\|\nabla\tilde{n}\|_{L^2}^2\\
&\leq\;C\|u_1\|_{L^\infty}\int_{\mathbb{R}^3}(|\tilde{n}|^2+|\nabla\tilde{n}|^2)dx
\end{aligned}$$

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$$+ C\|\nabla n_2\|_{L^\infty}\int_{\mathbb{R}^3}(|\tilde{n}|^2+|\tilde{u}|^2)dx$$
$$+ \; C\|n_2\|_{L^\infty}\int_{\mathbb{R}^3}(|\nabla\tilde{n}|^2+|\nabla\tilde{c}|^2)dx$$
$$+ C\|\nabla c_1\|_{L^\infty}\int_{\mathbb{R}^3}(|\tilde{n}|^2+|\nabla\tilde{n}|^2)dx$$
$$+ \tau\|(n_1+n_2)\|_{L^\infty}\int_{\mathbb{R}^3}|\tilde{n}|^2dx. \qquad (3.27)$$

For the estimate of $\tilde{u}$, multiplying $\tilde{u}$ to both sides of the second equation of (3.26) and taking integrations in x, we obtain

$$\int_{\mathbb{R}^3}\tilde{u}\,.\,\partial_t\tilde{u}dx-\int_{\mathbb{R}^3}\tilde{u}\,.\,\Delta\tilde{u}dx$$
$$= -\int_{\mathbb{R}^3}\tilde{u}\,.\,\nabla\tilde{\pi}dx \; -\int_{\mathbb{R}^3}\tilde{u}\,.\,(\tilde{u}\nabla\cdot u_1)dx$$
$$-\int_{\mathbb{R}^3}\tilde{u}\,.\,(u_2.\,\nabla\tilde{u})dx-\int_{\mathbb{R}^3}\tilde{u}\,.\,(\tilde{n}\nabla\varphi)dx.$$

By using integration by parts and the Cauchy inequality, we have

$$\frac{d}{2\,dt}\|\tilde{u}\|_{L^2}^2+\|\nabla\tilde{u}\|_{L^2}^2\le C\|\nabla\cdot u_1\|_{L^\infty}\|\tilde{u}\|_{L^2}^2+$$
$$C\|u_2\|_{L^\infty}(\|\tilde{u}\|_{L^2}^2+\|\nabla.\,\tilde{u}\|_{L^2}^2)+C\|\nabla\varphi\|_{L^\infty}\left(\|\tilde{u}\|_{L^2}^2\;+\|\tilde{n}\|_{L^2}^2\right).$$

Since L^∞ norms of σ^i, u^i, c^i where i = 1, 2 are bounded, we have

$$\frac{d}{2\,dt}\|\tilde{u}\|_{L^2}^2+C\|\nabla\tilde{u}\|_{L^2}^2\le C\|\tilde{n}\|_{L^2}^2+C\|\tilde{u}\|_{L^2}^2. \qquad (3.28)$$

Similarly, as above, we estimate $\tilde{c}$ as follows:

$$\frac{d}{2\,dt}\|\tilde{c}\|_{L^2}^2+C\|\nabla\tilde{c}\|_{L^2}^2+C\|\tilde{c}\|_{L^2}^2$$
$$\le C\left(\|\tilde{c}\|_{L^2}^2+\|\tilde{n}\|_{L^2}^2+\|\tilde{u}\|_{L^2}^2\right). \qquad (3.29)$$

Similarly, as above, we estimate $\tilde{v}$ as follows:

$$\frac{d}{2\,dt}\|\tilde{v}\|_{L^2}^2+C\|\nabla\tilde{v}\|_{L^2}^2+C\|\tilde{v}\|_{L^2}^2$$
$$\le C\left(\|\tilde{v}\|_{L^2}^2+\|\tilde{n}\|_{L^2}^2+\|\tilde{u}\|_{L^2}^2\right). \qquad (3.30)$$

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Then, after taking the linear combination of all estimates, we obtain

$$\frac{d}{2\,dt}\left(\|\tilde{n}\|_{L^2}^2+\|\tilde{u}\|_{L^2}^2+\|\tilde{c}\|_{L^2}^2+\|\tilde{v}\|_{L^2}^2\right)$$
$$+\mathrm{C}_1\left(\|\tilde{n}\|_{L^2}^2+\|\tilde{c}\|_{L^2}^2+\|\tilde{v}\|_{L^2}^2\right)$$
$$+\mathrm{C}_2\left(\|\nabla\tilde{n}\|_{L^2}^2+\|\nabla\tilde{u}\|_{L^2}^2+\|\nabla\tilde{c}\|_{L^2}^2+\|\nabla\tilde{v}\|_{L^2}^2\right)$$
$$\leq \mathrm{C}\left(\|\tilde{n}\|_{L^2}^2+\|\tilde{u}\|_{L^2}^2+\|\tilde{c}\|_{L^2}^2+\|\tilde{v}\|_{L^2}^2\right). \quad (3.31)$$

By applying Gronwall's inequality to the above equation, we have

$$\sup_{0\leq t\leq T_1}\left(\|\tilde{n}\|_{L^2}^2+\|\tilde{u}\|_{L^2}^2+\|\tilde{c}\|_{L^2}^2+\|\tilde{v}\|_{L^2}^2\right)$$
$$\leq e^{cT_1}\left(\|\tilde{n}(0)\|_{L^2}^2+\|\tilde{u}(0)\|_{L^2}^2+\|\tilde{c}(0)\|_{L^2}^2\right.$$
$$\left.+\|\tilde{v}(0)\|_{L^2}^2\right).$$

Since the initial data of ($\tilde{n}$,$\tilde{u}$, $\tilde{c}$, $\tilde{v}$) are all zero for T > 0, that implies the uniqueness of the local solution.

Conclusion

In this paper, We have established the local existence and uniqueness of classical solutions for the three-dimensional Keller–Segel–Navier–Stokes system with nonlinear diffusion and logistic source. We showed the existence of unique solutions by the energy method. We used integral by parts, Cauchy –Schwarz inequality, and Gronwall's inequality to prove the local existence and uniqueness of solutions.

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